Bipartitive families and the bi-join decomposition

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Abstract

This article introduces a new generalization of the modular decomposition called the *bi-join decomposition*. Like the *split decomposition*, it is an application of *bipartitive families* that are presented here. We show that the bi-join decomposition is unique, can be computed in linear time, and we give several characterizations for graphs completely decomposable by this decomposition.

1 Introduction

Graph decompositions are widely used in graph algorithms and graph theory. Among many well-known examples this paper deals with the modular decomposition, and one of its generalization, namely the split decomposition. Both correspond to a decomposition tree, whose leaves are the graph vertices and whose internal nodes correspond to decomposition operations. When the degree of the tree is bounded, some NP-hard problems can be solved in linear-time using induction on the tree [16]. Modular decomposition has been studied for a long time and can be computed in linear time [7, 15]. Some properties of the modular decomposition extend to the split decomposition, including unicity [8] and linear-time computation of the decomposition tree [11].

For a decomposition $\mathcal{D}$ we say that a graph G is *completely decomposable* if every subgraph (large enough, i.e with at least four or five vertices, depending on the decomposition) admits a non-trivial decomposition, and on the other side G is *prime* if it admits no non-trivial decomposition. The class of cographs is the class of graph completely decomposable by the modular decomposition [5]. The corresponding class for the split decomposition is the class of distance hereditary graphs [13]. Several characterizations are known for the class of cographs [5], and for the class of distance hereditary graphs [2, 13], including a characterization by forbidden subgraphs and by vertex extensions.

The bi-join decomposition is, like the split decomposition, an application of the *bipartitive families* that are presented here. Like [4] generalized modular decomposition to *partitive families*, set families with the same closure properties than module family, the bipartitive families are families of partition of a set in two classes that respect some closure properties. Similar results were found by [8, 9], working on extending split decomposition, but using a different formalism than the one presented here.

In this paper, we define bipartitive families and we show that they can be uniquely represented by a unrooted tree. We introduce a new generalization of the modular decomposition called the *bi-join decomposition*. We show that it leads to a decomposition tree, using the bipartitive families algebra. Finally we characterize the completely decomposable graphs, and we give a linear-time decomposition algorithm.

2 Graph decomposition background

2.1 Basic definitions

A graph is a pair $G = (V, E)$. We only consider finite undirected graphs without loop and without multiple edges. Let $n = |V|$ and $m = |E|$. We denote by $N_G(v) = \{u \in V : \{u, v\} \in E\}$ the *neighborhood* of v in G and by $N_G[v] = N_G(v) \cup \{v\}$ its *closed neighborhood*, and by $d_G(v) = |N_G(v)|$ the *degree* of v in G. We shall write $N(v)$, $N[v]$ and $d(v)$ if there is no ambiguity. Given $V' \subseteq V$, $G[V'] = (V', \{\{u, v\} \in E : u, v \in V'\})$ is the *subgraph* of G *induced* by V'. We denote by $G - V' = G[V \setminus V']$ the subgraph induced by $V \setminus V'$ and, if $v \in V$, we write $G - v$ instead of $G - \{v\}$. We say that a graph is *H-free* if it does not have H as induced subgraph, and *(H_1,...,H_k)-free* if it is H_i-free for all $i \in \{1, \ldots, k\}$.

A *path* between v_1 and v_k in a graph is a sequence of vertices $(v_1, \ldots, v_k)$ such that for all $i \in \{1, \ldots, k-1\}$, $\{v_i, v_{i+1}\} \in E$. k is the *length* of the path. A path is *chordless* if for all $0 < i < j \leq k$, $\{v_i, v_j\} \in E$ if and only if $i = j - 1$. A *cycle* in a graph is a sequence of vertices $(v_1, \ldots, v_k)$ such that $\{v_1, v_k\} \in E$ and for all $i \in \{1, \ldots, k-1\}$, $\{v_i, v_{i+1}\} \in E$. A cycle is *chordless* if for all $0 < i < j \leq k$, $\{v_i, v_j\} \in E$ if and only if $i = j - 1$ or $i = 1$ and $j = k$. An *hole* is a chordless cycle with at least 5 vertices. Figure 1 show some graphs discussed in this paper.

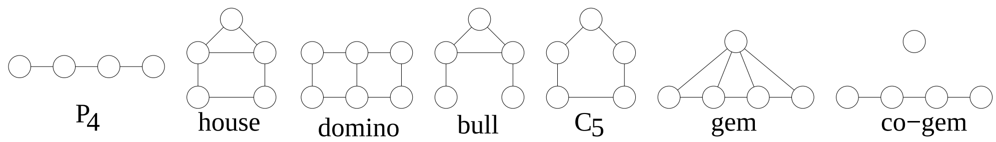

Figure 1: *Some graphs*

A graph is *connected* if there is a path between u and v for all vertices u and v. A *tree* is a connected graph without cycles. We shall denote vertices of tree by *nodes*. A node in a tree is a *leaf* if its degree is at most one, and is an *internal node* otherwise. A *rooted tree* $T = (V, E)$ is a tree with a *root* $r \in V$. For all $u \neq r$, the parent of u in T is the unique v such that v is adjacent to u, and v is on the path between u and r. A neighbor w of u which is not the parent of u is a *child*.

Let u be a vertex of G. u is a *pending vertex* of v if $N(u) = \{v\}$, a *false twin* of v if $N(u) = N(v)$ and a *true twin* of v if $N[u] = N[v]$. An *extension* of a graph G at a vertex u adds to G a new vertex v that can be twin of u or pending at u.

2.2 Modular decomposition

A *module* M of a graph $G = (V, E)$ is a non-empty set of vertices such that a vertex $v \notin M$ is adjacent either to all vertices of M or to none of them. Every singleton $\{v\}$ and the vertex set V are modules of any graph, called *trivial* modules. A graph is *prime w.r.t modular decomposition* if all its module are trivial. On the other side, a graph is *completely decomposable w.r.t modular decomposition* if every induced subgraph of at least 3 vertices contains a non-trivial module. The following theorem give several characterizations of the class of *cographs*.

Theorem 1. *[5]. The following propositions are equivalent:*

1. *G is a cograph.*
2. *G is completely decomposable w.r.t. modular decomposition.*
3. *G is P_4-free.*
4. *For every subgraph H of G, either H or its edge-complement $\overline{H}$ are not connected.*
5. *G can be obtained from a single vertex by a sequence of extensions by a true twin or a false twin.*

Two set A and B *overlap* if none of $A \setminus B$, $B \setminus A$, and $A \cap B$ is empty. A *strong module* is a module that does not overlap any other module. Given two strong modules, either they do not intersect or one of them contains the other one. Thus, the strong modules can be ordered into a tree by the inclusion relation. This tree is called *modular decomposition tree*. Its root is the strong module V, and its leaves are the n singletons $\{v\}$. Since every internal node has at least two sons, the tree has less than n internal nodes, and thus there are at most $n-2$ non-trivial strong modules in a graph.

The *sons* of a strong module M are the sons of M in the modular decomposition tree; in other words they are the largest strong modules included in M. According to [4], a strong module is *complete* if it has two sons A and B such that $A \cup B$ is a module, and otherwise is *prime*. [4, 16] made that fundamental observation:

- Every union of sons of a complete module is a module.
- Every module is either strong or is the union of some sons of a complete module.

The modular decomposition is the family of all modules of a graph [16]. There can be up to 2^n modules in a graph (for instance, in the complete graph) but the modular decomposition tree allows to store them in $O(n)$ space, using the fundamental observation above. This result is more formally established in 3.1.

2.3 Split Decomposition

In a graph, there is a *complete join* between $A \subseteq V$ and $B \subseteq V$ if every vertex of A is linked with every vertex of B. A *bipartition* of a set V is an (unordered) pair $\{V_1, V_2\}$ such that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. A *split* in a graph $G = (V, E)$ is a bipartition $\{V_1, V_2\}$ of V such that there exists $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ with a complete join between W_1 and W_2, and with no other edges between V_1 and V_2. If X is a module then $\{X, V \setminus X\}$ is a split, and thus the splits are proper generalizations of modules. A split is trivial if $|V_1| = 1$ or $|V_2| = 1$. A graph G is *prime w.r.t. split decomposition* if all its split are trivial, and is *completely decomposable w.r.t. split decomposition* if every induced subgraph of G with at least four vertices admits a non-trivial split.

The following theorem give several characterization of the class of *distance hereditary graphs*.

Theorem 2. *[2, 13] The following propositions are equivalent:*

1. *G is a distance hereditary graph.*

2. *G is completely decomposable w.r.t. split decomposition.*

3. *all chordless paths between any two vertices of G have the same length.*

4. *G is (house,hole,domino,gem)-free.*

5. *G can be obtained from a single vertex by a sequence of extensions by a true twin or a false twin or a pending vertex.*

There may be up to 2^n splits in a graph but, as we will see in 3.2, an unrooted tree can store them in $O(n)$ space. This less known result can be deduced from [9] and rests on the notion of *strong splits*.

3 Set families and bipartition families

3.1 Partitive families

We denote by $X \Delta Y$ the set $(X \setminus Y) \cup (Y \setminus X)$.

Definition 3. Let V be a set. A family $\mathcal{F}$ of subsets of V is a *rooted tree-like* family if:

1. $\emptyset \notin \mathcal{F}$ and $V \in \mathcal{F}$,

2. for all $v \in V$, $\{v\} \in \mathcal{F}$,

3. for all X and Y in $\mathcal{F}$, X and Y do not overlap.

The *inclusion tree* of a rooted tree-like family is a rooted tree T with node set $\mathcal{F}$, with root V and edge set $\{\{X, Y\} : X \subsetneq Y$ and there is no $Z \in \mathcal{F}$ such that $X \subsetneq Z \subsetneq Y\}$.

Definition 4. Let V be a set. A family $\mathcal{F}$ of subsets of V is *partitive* if:

1. $\emptyset \notin \mathcal{F}$ and $V \in \mathcal{F}$,

2. for all $v \in V$, $\{v\} \in \mathcal{F}$,

3. for all X and Y in $\mathcal{F}$ such that X and Y overlap, then

    - $X \cap Y \in \mathcal{F}$,
    - $X \cup Y \in \mathcal{F}$,
    - $X \setminus Y \in \mathcal{F}$,

- $Y \setminus X \in \mathcal{F}$ and
- $X \Delta Y \in \mathcal{F}$.

Note that a rooted tree-like family is a partitive family. A member X of a partitive family $\mathcal{F}$ is *strong* if there is no member $Y \in \mathcal{F}$ such that X and Y overlap. The strong members of a partitive family form a rooted tree-like family. The following theorem show that the inclusion tree of the strong members of a partitive family can be used as a representation of all members of the family:

Theorem 5. *[4] Let $\mathcal{F}$ be a partitive family and let T be the inclusion tree of the strong members of $\mathcal{F}$. The nodes of T can be labeled* complete *or* prime *in such a way that:*

- *every union of children of a complete node is a member of $\mathcal{F}$*
- *for all member X of $\mathcal{F}$ which is not strong, there is a complete node α such that X is a union of children of α.*

We call this tree the *representative tree* of the family. As seen before, the modules of a graph form a partitive family, and the representative tree of such a family is called modular decomposition tree. An other example is given by the disjunctive decomposition of switching functions [1].

3.2 Bipartitive families

Two bipartitions of V $\{V_1, V_2\}$ and $\{V_3, V_4\}$ *overlap* if the four sets $V_1 \cap V_3$, $V_2 \cap V_3$, $V_1 \cap V_4$ and $V_2 \cap V_4$ are nonempty. A bipartition $\{V_1, V_2\}$ is *trivial* if $|V_1| = 1$ or $|V_2| = 1$.

Let $T = (V_T, E_T)$ be a tree with leaf-set $V \subseteq V_T$, and e an edge of T. $T' = (V_T, E_T \setminus \{e\})$ has two connected components T_e^1 and T_e^2. For $i \in \{1, 2\}$, let C_e^i be the set of nodes in T_e^i which are leaves in T. Every edge e of T thus defines a bipartition $\{C_e^1, C_e^2\}$ of V. Let α be an internal node of T. $T - \alpha$ has $d(\alpha)$ connected components $T_\alpha^1, \ldots, T_\alpha^{d(\alpha)}$. For all $i \in \{1, \ldots, d(\alpha)\}$, we denote by C_α^i the set of nodes in T_α^i which are leaves in T. $\{C_\alpha^1, \ldots, C_\alpha^{d(\alpha)}\}$ is a partition of V.

Definition 6. Let V be a set. A family $\mathcal{F}$ of bipartitions of V is a *unrooted tree-like* family if:

1. $\{\emptyset, V\} \notin \mathcal{F}$
2. for all $v \in V$, $\{\{v\}, V \setminus \{v\}\} \in \mathcal{F}$,

3. two bipartitions of $\mathcal{F}$ do not overlap.

A *bipartition tree* of a unrooted tree-like family is a unrooted tree T with leaf set V such that for each bipartition $\{V_1, V_2\}$ of $\mathcal{F}$ there is an edge e of T such that $C_e^1 = V_1$ and $C_e^2 = V_2$, and conversely for every edge e $\{C_e^1, C_e^2\} \in \mathcal{F}$.

Lemma 7. *Let $\mathcal{F}$ be a unrooted tree-like family with bipartition tree T, and let $\{V_1, V_2\}$ be a bipartition of V such that $\{V_1, V_2\}$ does not overlap any member of $\mathcal{F}$. Then there is a unique node α of T such that for all $a \in \{1, \ldots, d(\alpha)\}$, $C_\alpha^a \subseteq V_1$ or $C_\alpha^a \subseteq V_2$.*

Proof. Let $e = \{\beta, \gamma\}$ be a edge of T. By definition $\{C_e^1, C_e^2\} \in \mathcal{F}$. $\{V_1, V_2\}$ does not overlap $\{C_e^1, C_e^2\}$, thus there is $x, y \in \{1, 2\}$ such that $C_e^x \cap V_y = \emptyset$. W.l.o.g. suppose that $x = 1$, and that β is the node such that there is a $a \in \{1, \ldots, d(\beta)\}$ with $C_\beta^a = C_e^2$ (so there is a $b \in \{1, \ldots, d(\gamma)\}$ with $C_\gamma^b = C_e^1$). Let us *direct* e from β to γ. Every edges of T can be directed in such a way.

Let f be a edge of T on the C_e^1 side (nearer from β than from γ). It corresponds to a bipartition $\{C_f^1, C_f^2\}$ of $\mathcal{F}$. There exists $z \in \{1, 2\}$ such that $C_f^z \subseteq C_e^1$, therefore $C_f^z \cap V_y = \emptyset$. The edge f is thus directed toward β.

We proved that in the directed tree, for every two directed edges e and f, if e is directed in the direction opposed to f then f is directed toward e. T is thus a directed intree and admits an unique root α which has the desired property. $\square$

Theorem 8. *Each unrooted tree-like family admits one and only one bipartition tree*

Proof. We construct the bipartition tree using induction on the non-trivial bipartitions. Let $\mathcal{F}$ be a unrooted tree-like family and $\{V_1^1, V_2^1\}, \ldots, \{V_1^k, V_2^k\}$ be its non-trivial bipartitions. For each $0 \leq i \leq k$ let $\mathcal{F}_i$ be the family containing the trivial bipartitions plus bipartitions $\{V_1^1, V_2^1\}, \ldots, \{V_1^i, V_2^i\}$. It is clearly an unrooted tree-like family. We construct recursively the bipartition tree T_i of $\mathcal{F}_i$.

The trivial unrooted tree-like family $\mathcal{F}_0$ contains just the trivial bipartition; its partition tree T_0 is a star with n leaves and one central node. Suppose that T_i is a bipartition tree of $\mathcal{F}_i$. Lemma 7 says that there is a unique node α in T_i such that for all $a \in \{1, \ldots, d(\alpha)\}$ there is $b \in \{1, 2\}$ such that C_α^a is included in V_b^{i+1}. Then the tree T_{i+1} is built from T_i by splitting this node α into two new nodes : α_1 contains the parts C_α^b that are included in V_1^{i+1} and α_2 the parts included in V_2^{i+1}. The two nodes are linked by an edge e and we have $\{V_1^{i+1}, V_2^{i+1}\} = \{C_e^1, C_e^2\}$. Then T_{i+1} is uniquely determined from T_i and $\{V_1^{i+1}, V_2^{i+1}\}$. At least, T_k is the bipartitive tree of $\mathcal{F}_k = \mathcal{F}$ and is unique. $\square$

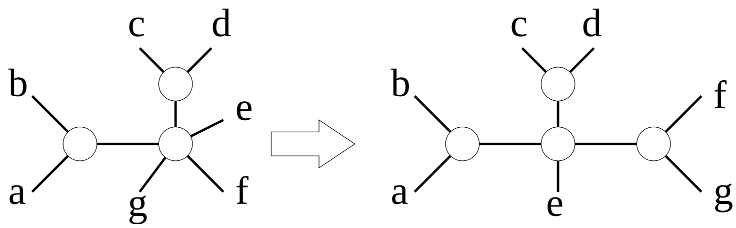

Figure 2: *Left : bipartition tree for the family with non-trivial strong members $\{ab, cdefg\}$ and $\{abefg, cd\}$. Right : non-trivial member $\{abcde, fg\}$ is added. Notice there is exactly one edge per strong members, including the trivial ones.*

Furthermore, the proof above can lead to a $O(kn)$-time algorithm that builds the bipartition tree: at each insertion of a new bipartition, the node α can be identified by marking all leaves from V_1^{i+1} and recursively marking every internal node whose all sons excepted one are marked. When no more node can be marked, if only one vertex is neighbor of both marked and unmarked nodes, then it is split into its marked and unmarked neighbours. And if many vertices are in that case, then $\{V_1^{i+1}, V_2^{i+1}\}$ overlaps some previously inserted bipartition: the algorithm is robust.

Definition 9. Let V be a set. A family $\mathcal{F}$ of bipartitions of V is *bipartitive* if:

1. $\{\emptyset, V\} \notin \mathcal{F}$,
2. for all $v \in V$, $\{\{v\}, V \setminus \{v\}\} \in \mathcal{F}$,
3. for all $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ in $\mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap, then
   - $\{X_1 \cap Y_1, X_2 \cup Y_2\}$,
   - $\{X_1 \cap Y_2, X_2 \cup Y_1\}$,
   - $\{X_2 \cap Y_1, X_1 \cup Y_2\}$,
   - $\{X_2 \cap Y_2, X_1 \cup Y_1\}$ and
   - $\{X_1 \Delta Y_1, X_1 \Delta Y_2\}$ are in $\mathcal{F}$.

A member $\{X_1, X_2\}$ of a bipartitive family $\mathcal{F}$ is *strong* if there is no member $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap. Note that the trivial bipartitions are strong. The strong members clearly form a unrooted tree-like family.

Let T be a partition tree. For each node α with degree k and each $\emptyset \subsetneq I \subsetneq \{1, \dots, k\}$, α and I define the bipartition $\{V_I, V \setminus V_I\}$ where $V_I = \bigcup_{i \in I} C_\alpha^i$. Let $\mathcal{B}(T, \alpha)$ be the family $\{\{V_I, V \setminus V_I\}\}_{\emptyset \subsetneq I \subsetneq \{1,\dots,k\}}$ of all possible bipartition get by splitting node α.

The following theorem can be found in [8, 9], written using a different formalism (the *simple decomposition* paradigm instead of *bipartitions* paradigm). We prove it here more concisely.

Theorem 10. *Let V be a set and $\mathcal{F}$ be a bipartitive family of subsets of V, and let T be the partition tree of its strong bipartitions. The nodes of T can be labeled* complete *or* prime *such that:*

1. *every bipartition of $\mathcal{F}$ is either strong or belongs to $\mathcal{B}(T, \alpha)$ for some complete node α of T.*

2. *for every complete node α of T, $\mathcal{B}(T, \alpha) \subseteq \mathcal{F}$*

Proof. Let $S(\mathcal{F})$ be the subfamily of strong bipartitions of $\mathcal{F}$ and T its bipartition tree.

1. Let B be a bipartition of $\mathcal{F}$ that is not strong. B overlaps no bipartition of $S(\mathcal{F})$ since every member of $S(\mathcal{F})$ is strong. According to Lemma 7 there is a unique node α of T such that B belongs to $\mathcal{B}(T, \alpha)$.

2. Let α be a complete node of T and $\{C_\alpha^i\}_{i \in \{1\dots,d(\alpha)\}}$ the partition of V that the connected components of $T - \alpha$ induce.

For short we will denote $B(I)$, with $I \subseteq \{1, \dots, d(\alpha)\}$, the bipartition $\{V_I, V \setminus V_I\}$ where $V_i = \bigcup_{i \in I} C_\alpha^i$, and for few indices note $B(i, j)$ instead of $B(\{i, j\})$.

Claim. For each $i, j \in \{1, \dots, d(\alpha)\}$, $B(i, j) \in \mathcal{F}$.

Proof. First let us suppose that there exists no $I \subseteq \{1, \dots, d(\alpha)\}$ such that $i \in I$, $j \in I$ and $B(I) \in \mathcal{F}$. Let us consider the family $F = \{I_1, \dots, I_k\}$ such that for all $1 \leq j \leq k$, $I_j \subseteq \{1, \dots, d(\alpha)\}$, and $i \in I_j$, and $B(I_j) \in \mathcal{F}$. Since α is complete there exists at least one $B(I_0) \in \mathcal{F}$. If $i \notin I_0$ take $I_1 = \{1, \dots, d(\alpha)\} \setminus I_0$: $B(I_1) = B(I_0)$. F is therefore not empty.

Let F' be the subfamily of F containing elements of F maximal w.r.t. inclusion. Two elements I_1 and I_2 of F' overlap, therefore according to Definition 9, $B(I_1 \cup I_2) \in \mathcal{F}$. Let I' be the union of all elements of F'. Then $B(I') \in \mathcal{F}$.

Let us suppose there exists $B \in \mathcal{F}$ that overlaps $B(I')$. There exists J such that $B = B(J)$. We can suppose $i \in J$ (otherwise take $J' = \{1, \ldots, d(\alpha)\} \setminus J$). But $B(I' \cup J) \in \mathcal{F}$ according to Definition 9, and $I' \cup J$ is larger than I', contradiction since it is not in F'. Therefore $B(I')$ is strong.

As we suppose that there is no $I \subseteq \{1, \ldots, d(\alpha)\}$ such that $i \in I$, $j \in I$ and $B(I) \in \mathcal{F}$, then I' is an union of sets which do not contain j, and I' does not contain j. Therefore $B(I')$ is a non-trivial strong bipartition of $\mathcal{F}$. But there is no edge e in T such that $B(I') = \{C_e^1, C_e^2\}$, contradiction.

We can now suppose there exists $I \subseteq \{1, \ldots, d(\alpha)\}$ such that $i \in I$, $j \in I$ and $B(I) \in \mathcal{F}$. Let us take I with minimum size. $B(I)$ is not strong, thus there exists $B \in \mathcal{F}$ that overlaps $B(I)$, and there exists J such that $B = B(J)$. If neither i nor j is in J then $B(I \setminus J)$ is a bipartition of $\mathcal{F}$ that contradicts minimality of I. If both i and j are in J then $B(I \cap J)$ is a bipartition of $\mathcal{F}$ that contradicts minimality of I. So J contains exactly one of i or j.

Let us suppose I contains a third element k. If there exists J_1 containing i and k, and J_2 containing j and k, such that $B(J_1)$ and $B(J_2)$ are two bipartitions of $\mathcal{F}$ that overlap $B(I)$, then according to definition 9, $B(I \cap J_1) \in \mathcal{F}$, and $B(I \cap J_2) \in \mathcal{F}$, and finally $B((I \cap J_1)\Delta(I \cap J_2)) \in \mathcal{F}$. But i and j are belong to $(I \cap J_1)\Delta(I \cap J_2)$, and not k which contradicts minimality of I.

Therefore either all bipartitions $B(J)$ that overlaps $B(I)$ are such that $\{i, k\} \subseteq J$, or all bipartitions $B(J)$ that overlaps $B(I)$ are such that $\{j, k\} \subseteq J$. In first case $B(i, k)$ is strong, and in second case $B(j, k)$ is strong, but they have no edge in T, as they belong to $\mathcal{B}(T, \alpha)$, contradiction. So $I = \{i, j\}$. $\square$

Then for all $I \subseteq \{1, \ldots, d(\alpha)\}$, we pose $I = \{i_1, i_2, \ldots, i_k\}$. For all $1 \leq j < k$, $B(i_j, i_{j+1}) \in \mathcal{F}$. Let $I_j = \{i_1, \ldots, i_j\}$. For all $1 < j < k$, $B(I_j)$ overlaps $B(i_j, i_{j+1})$. If we suppose $B(I_j) \in \mathcal{F}$, then according to definition 9, $B(I_{j+1}) \in \mathcal{F}$. Notice $B(I_2) = B(i_1, i_2) \in \mathcal{F}$. So by recurrence $B(I_k) = B(I) \in \mathcal{F}$. $\square$

The partition tree with the *complete* or *prime* labels will be called the *bipartitive tree* of the family $\mathcal{F}$. It is an $O(n)$-sized representation of $\mathcal{F}$.

The family of all split in a connected graph is a bipartitive family, and the *split decomposition tree* is the representative tree of the family of split [8].

To conclude this section, the following lemma show a strong relation between the partitives families and the bipartitive families.

Lemma 11. *Let V be a set and $v \in V$. Let $\mathcal{F}$ be a family of bipartition of V, and $\mathcal{F}' = \{\ V' : v \notin V' \text{ and } \{V', V \setminus V'\} \in \mathcal{F}\}$. Then $\mathcal{F}$ is bipartitive if and only if $\mathcal{F}'$ is partitive.*

In this case, let T' be the representative tree of $\mathcal{F}'$. Then the representative tree T of $\mathcal{F}$ is isomorph to T' with a additional leaf for the vertex v adjacent to the root of T'. Moreover, an internal node of T is complete (prime) if and only if the corresponding node is complete (prime) in T'.

Proof. $\mathcal{F}$ fulfills condition (1) and condition (2) in the definition 9 if and only if $\mathcal{F}'$ fulfill condition (1) and (2) in definition 4. It suffice to show that the condition (3) in the definition 4 is fulfilled by $\mathcal{F}$ if and only if the condition (3) in the definition 9 is fulfilled by $\mathcal{F}'$. Let $\{X, V \setminus X\}$ and $\{Y, V \setminus Y\}$ be two bipartition of V. W.l.o.g assume $v \notin X$ and $v \notin Y$. According to the definition, if two bipartitions $\{X, V \setminus X\}$ and $\{Y, V \setminus Y\}$ overlap then X and Y overlap. Conversely if X and Y overlap, the bipartitions must overlap since $v \in (V \setminus X)$ and $v \in (V \setminus Y)$: the two set $V \setminus X$ and $V \setminus X$ also overlap.

The two bipartitions $\{X, V \setminus X\}$ and $\{Y, V \setminus Y\}$ thus overlap if and only if the two sets X and Y overlap. In this case, the bipartitions $\{X \cap Y, V \setminus (X \cap Y)\}$, $\{X \cup Y, V \setminus (X \cup Y)\}$, $\{X \setminus Y, V \setminus (X \setminus Y)\}$, $\{Y \setminus X, V \setminus (Y \setminus X)\}$ and $\{X \Delta Y, V \setminus (X \Delta Y)\}$ are in $\mathcal{F}$ if and only if $X \cap Y$, $X \cup Y$, $X \setminus Y$, $Y \setminus X$ and $X \Delta Y$ are in $\mathcal{F}'$.

For the second part, we show that the tree T is the representative tree of $\mathcal{F}$. Let $\{X, V \setminus X\}$ be a bipartition of V such that $v \notin X$. The bipartition $\{X, V \setminus V\}$ is a strong member of $\mathcal{F}$ if and only if X is a strong member of $\mathcal{F}'$. In this case, there is a node α for X in T'. If $X \neq V \setminus \{v\}$, the bipartition $\{X, V \setminus X\}$ is the bipartition $\{C_e^1, C_e^2\}$ for the edge e between α and its parent in T'. If $X = V \setminus \{v\}$, then $\{X, V \setminus X\}$ is the bipartition $\{C_e^1, C_e^2\}$ for the edge e we added in T between the root and the leaf for v.

Note that a node α in the representative tree of $\mathcal{F}$ is complete if and only if there is a bipartition $\{X, V \setminus X\} \in \mathcal{F}$ and $\emptyset \subsetneq I \subsetneq \{1, \ldots, d(\alpha)\}$ such that $X = \cup_{i \in I} C_\alpha^i$. W.l.o.g. $v \notin X$. Such a partition exists if and only if $X \in \mathcal{F}'$, and thus α is complete in T'. $\square$

4 Bi-joins and the bi-join decomposition

Definition 12. A *bi-join* in a graph $G = (V, E)$ is a bipartition $\{V_1, V_2\}$ of V such that there exists $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ with a complete join between W_1 and W_2, a complete join between $V_1 \setminus W_1$ and $V_2 \setminus W_2$, and no other edges between V_1 and V_2.

A bi-join $\{X_1, X_2\}$ is *strong* if it is a strong member of the family of bi-joins (*i.e.* there is no bi-join $\{Y_1, Y_2\}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap), and is *trivial* if $|V_1| = 1$ or $|V_2| = 1$. A graph is *prime w.r.t the bi-join decomposition* if all its bi-join are trivial.

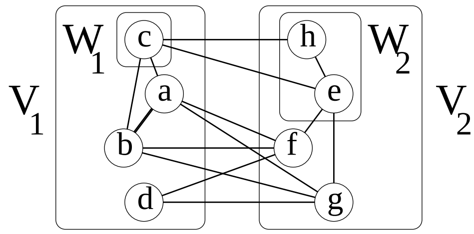

Figure 3: *A bi-join of a graph.*

Theorem 13. *[17] The family of bi-joins of a graph is a bipartitive family.*

The proof of this theorem is given below. The *bi-join decomposition tree* of a graph G is the representative tree of the bipartitive family of all bi-joins of G. As corollary of theorem 10 and theorem 13, we obtain:

Corollary 14. *There is a unique tree T, call the* bi-join decomposition tree*, whose leaf set is the vertex set of G, where nodes are labeled* complete *or* prime*, with the following properties. For every bi-join $\{V_1, V_2\}$ of G there is a node α in T such that V_1 is the union of components of α. Conversely if α has components $C_1, \dots, C_{d(\alpha)}$, if α is labeled* complete *then any union of components is a bi-join (for all $I \subseteq \{1, \dots, d(\alpha)\}$, $\{\cup_{i\in I} C_i, \cup_{i\notin I} C_i\}$ is a bi-join) and if α is labeled* prime *then for every $a \in \{1, \dots, d(\alpha)\}$, $\{C_a, \cup_{i\neq a} C_i\}$ is a bi-join, and there is no other bi-join.*

Now let us proof Theorem 13. The *Seidel switch* [18] of a graph G with switch subset $W \subseteq V$ is a graph with same vertex-set and there is an edge between x and y if either $\{x, y\} \in E(G)$ and $\{x, y\}$ does not overlap W, or $\{x, y\} \notin E(G)$ and $\{x, y\}$ overlaps W. Let the *Seidel reduction* of a graph G at vertex v, denoted $\overline{G}^v$, be the removing of v from vertex-set after a Seidel switch with switch subset $N(v)$.

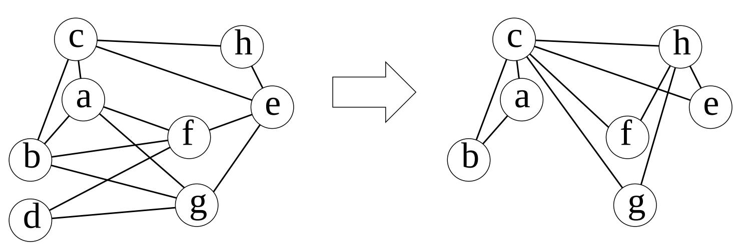

Figure 4: *The Seidel reduction at vertex d.*

Lemma 15 (Fundamental lemma). *Let $\{V_1, V_2\}$ be a bipartition of V and $v \in V_1$. Then $\{V_1, V_2\}$ is a bi-join of G if and only if V_2 is a module of $\overline{G}^v$.*

Proof. Let $\{V_1, V_2\}$ be a bipartition of G and let $v \in V_1$. V_2 is a module of $\overline{G}^v$ if and only if for all $w \in V_1 \setminus \{v\}$, $V_2 \subseteq N_{\overline{G}^v}(w)$ or $V_2 \cap N_{\overline{G}^v}(w) = \emptyset$. Moreover $\{V_1, V_2\}$ is a bi-join of G if and only if for all $w \in V_1 \setminus \{v\}$, $V_2 \cap N_G(w) = V_2 \cap N_G(v)$ or $V_2 \cap N_G(w) = V_2 \setminus N_G(v)$.

Suppose that v is adjacent to w. $V_2 \subseteq N_{\overline{G}^v}(w)$ if and only if for all $u \in V_2$, u is adjacent to both v and w, or u is adjacent to none of v and w. Then $V_2 \subseteq N_{\overline{G}^v}(w) \iff V_2 \cap N_G(w) = V_2 \cap N_G(v)$. Moreover, $V_2 \cap N_{\overline{G}^v}(w) = \emptyset$ if and only if for all $u \in V_2$, u is adjacent to exactly one of v and w. Then $V_2 \cap N_{\overline{G}^v}(w) = \emptyset \iff V_2 \cap N_G(w) = V_2 \setminus N_G(v)$.

If v is not adjacent to w, then the previous observation on $\overline{G}$ gives immediately $V_2 \subseteq N_{\overline{G}^v}(w) \iff V_2 \cap N_G(w) = V_2 \setminus N_G(v)$ and $V_2 \cap N_{\overline{G}^v}(w) = \emptyset \iff V_2 \cap N_G(w) = V_2 \cap N_G(v)$. In all cases $V_2 \cap N_{\overline{G}^v}(w) = \emptyset$ or $V_2 \subseteq N_{\overline{G}^v}(w)$ if and only if $V_2 \cap N_G(w) = V_2 \cap N_G(v)$ or $V_2 \cap N_G(w) = V_2 \setminus N_G(v)$. $\square$

This Fundamental Lemma and Lemma 11 allow to prove Theorem 13: the bi-joins form a bipartitive family since the modules form a partitive family. More precisely, let $\mathcal{F}$ be the family of all bi-joins of a graph $G = (V, E)$, let $v \in V$, and let $\mathcal{F}' = \{V' : v \notin V' \text{ and } \{V', V \setminus V'\} \in \mathcal{F}\}$. By lemma 11, $\mathcal{F}$ is bipartitive if and only if $\mathcal{F}'$ is partitive, and by lemma 15 $\mathcal{F}'$ is the family of all modules of $\overline{G}^v$ and thus is partitive.

Degenerated graphs

Definition 16. A graph is *degenerated* for the bi-join decomposition if every bipartition of its vertices is a bi-join.

Lemma 17. *The degenerated graphs are exactly the complete bipartite graphs $K_{a,b}$ and the union of two complete graphs $K_a \oplus K_b$.*

Proof. It is easy to verify that these graphs are degenerated. Conversely let G be a degenerated graph. For all $v \in V(G)$, $\overline{G}^v$ is a degenerated graph for modular decomposition, i.e. every subset of vertices of $\overline{G}^v$ is a module, according to the Fundamental Lemma. It is well-known that degenerated graph for modular decomposition are exactly the complete graph and the stable set.

If $\overline{G}^v$ is a complete graph then in G there is no edge between $N(v)$ and $V \setminus N(v)$. Furthermore $N(v)$ and $V \setminus N(v)$ are two cliques, thus $G = K_a \oplus K_b$. And if $\overline{G}^v$ is a stable set then in G there is a complete join between $N(v)$ and $V \setminus N(v)$. Furthermore $N(v)$ and $V \setminus N(v)$ are two stable sets, thus $G = K_{a,b}$. $\square$

The bi-join components

Let G be a graph, and let α be a node of its bi-join decomposition tree. For every $i \in \{1, \ldots, d(\alpha)\}$, we choose a vertex $v_i \in C^i_\alpha$. The *bi-join component* of α, denoted $BJC(\alpha)$, is the graph $G[\{v_1, \ldots, v_k\}]$. Note that $BJC(\alpha)$ depends from the choice of $v_1, \ldots, v_k$.

Lemma 18. *If α is complete $BJC(\alpha)$ is either the complete bipartite graph $K_{a,b}$ or the union of two complete graphs $K_a \oplus K_b$. If α is prime then $BJC(\alpha)$ is prime.*

Proof. If α is complete then by definition its bi-join component is degenerated, and thus is $K_{a,b}$ or $K_a \oplus K_b$ according to lemma 17.

Let a pair (A, B) such that $\{A, B\}$ is a bipartition of V. We denote by $\sim_{(A,B)}$ the relation on A such that $u \sim_{(A,B)} v$ if $N(u) \cap B = N(v) \cap B$ or if $\{N(u) \cap B, N(v) \cap B\}$ is a partition of B. Note that $\sim_{(A,B)}$ is a equivalence relation, and that $\{A, B\}$ is a bi-join if and only if $\sim_{(A,B)}$ has a unique equivalence class.

Let α be a prime node. Suppose that $BJC(\alpha)$ is not prime w.r.t. bi-join decomposition, and $\{A, B\}$ is non trivial bi-join of $BJC(\alpha) = G[\{v_1, \ldots, v_k\}]$. Let $V_1 = \cup_{v_i \in A} C^i_\alpha$. We show that $\{V_1, B\}$ is a bi-join of $G[V_1 \cup B]$. For every $i \in \{1, \ldots, k\}$ such that $v_i \in A$ and for every $u \in C^i_\alpha$, $u \sim_{(V_1,B)} v_i$, since $\{C^i_\alpha, V \setminus C^i_\alpha\}$ is a bi-join of G. Moreover, for every $v_i, v_{i'} \in A$, $v_i \sim_{(V_1,B)} v_{i'}$ since $\{A, B\}$ is a bi-join if $G[A \cup B]$. Then $\sim_{(V_1,B)}$ has a unique equivalence class and $\{V_1, B\}$ is a bi-join of $G[V_1 \cup B]$. Using the same argument, the bi-join $\{B, V_1\}$ of $G[V_1 \cup B]$ can be extended to a bi-join $\{V_2, V_1\}$ of G, with $V_2 = \cup_{v_i \in B} C^i_\alpha$. $\{V_1, V_2\}$ is not in the bi-join decomposition tree of G and we have a contradiction. Thus $BJC(\alpha)$ is prime w.r.t. bi-join decomposition. □

5 Completely decomposable graphs

In Section 2 were given characterizations of graphs completely decomposable w.r.t. modular decomposition (cographs) and w.r.t. split decomposition (distance hereditary graphs). Now we present a similar characterization for the completely decomposable graphs w.r.t. bi-join decomposition. There are two equivalent definition for a graph completely decomposable w.r.t. bi-join decomposition:

Lemma 19. *Let G be a graph. The following conditions are equivalent:*
(1) every induced subgraph G with at least 4 vertices has a non trivial bi-join,
(2) every node in the bi-join decomposition tree of G is complete.

Proof. Suppose that every node in the bi-join decomposition tree of G is complete. Let H be a subgraph of G with at least 4 vertices. Then every node in the bi-join decomposition tree T of H is complete. Let α be a node in T. If the degree of α at least 4, then the bipartition induced by two components of α is non-trivial and is a bi-join of H. Otherwise, there is a component C^i_α of size at least 2 since the degree of α is 3 and $V(H) \geq 4$. Then $\{C^i_\alpha, V(H) \setminus C^i_\alpha\}$ is non-trivial and is a bi-join of H.

Suppose that there is a node α in the bi-join decomposition tree of G which is prime. Then by lemma 18, $BJC(\alpha)$ is prime and is a induced subgraph of G. $\square$

Let u be a vertex of G. We recall that v is a *false twin* of u if $N(u) = N(v)$ and v is a *true twin* of u if $N(u) \cup \{u\} = N(v) \cup \{v\}$. We say that v is a *false anti-twin* of u if $N(u) = V \setminus (N(v) \cup \{u, v\})$ and a *true anti-twin* of u if $N(u) = V \setminus N(v)$. An *(twin,anti-twin)-extension* of the graph is to select a vertex u and add another vertex that can be a true or false twin or anti-twin of u. The following theorem gives equivalent characterizations for the class of graphs completely decomposable w.r.t. bi-join decomposition.

Theorem 20. *Let G be a graph. The following conditions are equivalent:*
(1) G is completely decomposable w.r.t. bi-join decomposition.
(2) G is (C_5,bull,gem,co-gem)-free.
(3) G can be obtained from a single vertex by a sequence of (twin,anti-twin)-extensions.
(4) $\forall v \in V$, $\overline{G}^v$ is a cograph.

Proof. (1) $\iff$ (4): By lemma 11 and lemma 15, every node in the bi-join decomposition tree of G is complete if and only if every node in the modular decomposition tree of $\overline{G}^v$ is complete. So G is completely decomposable by the bi-join decomposition if and only if $\overline{G}^v$ is a cograph.

(1) $\Rightarrow$ (3): By induction on the number of vertices of the graph. It is trivial for $|V| \leq 3$. Otherwise, there is a node α in the bi-join decomposition tree T adjacent to two leaves u and v. This node is complete so $\{\{u, v\}, V \setminus \{u, v\}\}$ is a bi-join of G. This bi-join corresponds to a true or a false twin or anti-twin. Using notation of Definition 12 if $W_1 = \{u, v\}$ or $W_1 = \emptyset$ then $\{u, v\}$ is a module and thus these two vertices are twin. And else a vertex $x \notin \{u, v\}$ is neighbor of either u or v and these two vertices are thus antitwins.

(3) $\Rightarrow$ (4): By induction on the number of vertices of the graph. Let G be a graph obtained from a single vertex by a sequence of (twin,anti-twin)-extensions, let $u \in V(G)$, and G' a graph obtained from G by a (twin,anti-twin)-extension: we add a new vertex w which is a true or a false twin or

anti-twin of u. Let $v \in V(G)$; by induction $\overline{G}^v$ is a cograph. If $u \neq v$, w is a twin of u in $\overline{G'}^v$, so $\overline{G'}^v$ is a cograph by theorem 1. If $u = v$ and w is a true twin or a false anti-twin of u, w is a dominating vertex of $\overline{G'}^v$ (a vertex adjacent to all others vertices). If $u = v$ and w is a false twin or a true anti-twin of u, w is a isolated vertex of $\overline{G'}^v$ (a vertex of degree zero). In all cases, $\overline{G'}^v$ is a cograph.

(2) $\iff$ (4): Let H be a graph and $v \in V(H)$. Then $\overline{H}^v$ is a P_4 if and only if H is a C_5, a bull, a gem or a co-gem.

Let G be a graph such that for every $v \in V$, $\overline{G}^v$ is a cograph. Let H be an induced subgraph of G, and let $v \in V(H)$. Then $\overline{H}^v = \overline{G}^v[V(H) \setminus \{v\}]$ and then is not a P_4, so H is not a C_5, a bull, a gem or a co-gem. Conversely, let G be a (C_5,bull,gem,co-gem)-free graph, and let $v \in V$. Let H' be a subgraph of $\overline{G}^v$. $G[V(H) \cup \{v\}]$ is not a C_5, a bull, a gem or a co-gem, so $H' = \overline{G[V(H) \cup \{v\}]}^v$ is not a P_4. $\square$

In [14], Hertz show that the class of (C_5,bull,gem,co-gem)-free graphs is exactly the class of graphs for which every Seidel switch is perfect. He also gives a proof for (2) $\iff$ (4), and a recognition algorithm with running time $O(n^2)$ using the cograph recognition algirithm given in [6]. Using observation given in section 6 and a linear time cograph recognition algorithm [3, 6, 12], this class of graph can be recognized in linear time.

6 Linear-time decomposition algorithm

Theorem 21. *The bi-join decomposition of a graph can be computed in linear time.*

Proof. It is a consequence of the Fundamental Lemma and of the existence of linear-time modular decomposition algorithms [7, 15]. By lemma 11 and lemma 15, the bi-join decomposition tree of G is the modular decomposition tree of $\overline{G}^v$ with an additional node for v adjacent to the root. Let n be the number of vertices and m be the number of edges of G. The number of edges of $\overline{G}^v$ is at most $m + d(v) \times (n - d(v) - 1) \leq m + d(v) \times n$. Then if we choose v of minimum degree, then the number of edges of $\overline{G}^v$ is at most three times the number of edges of G, and the modular decomposition tree of $\overline{G}^v$ can be computed in linear time. $\square$

7 Missing properties of bi-join decomposition

The modular decomposition can be built recursively : one can find the maximal strong modules $M_1, \ldots, M_k$ of a graph, and the re-launch recursively the process on $G[M_i]$ for each i. This constructs the modular decomposition tree top-down and is indeed how the first modular decomposition algorithm worked [10]. But this recursive approach does not work for bi-joins since if $\{V_1, V_2\}$ is a bi-join of a graph G, a bi-join $\{V_{11}, V_{12}\}$ of $G[V_1]$ can not in general be extended into a bi-join of G.

Another difference between modular decomposition and bi-join decomposition is that we were not able to define the *quotient* of a node for the bi-join decomposition. For the modular decomposition, the quotient of a node α of the modular decomposition tree having sons $\alpha_1, \ldots, \alpha_k$ is the graph $G[\{v_1, \ldots, v_k\}]$ where v_i is an arbitrary chosen vertex in the module corresponding to α_i. [16] proved that the quotient is uniquely defined, and that

- if α is a prime node its quotient is a prime graph
- if α is a complete node its quotient is degenerated (either a clique or a stable set).

We were able to prove a weaker version of this result, using the bi-join components : the bi-join component of a node is not unique (it depends from the choice of v_i in C_α^i) but has the same property of being prime or degenerated.

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